Nodal fully discrete polytopal scheme for mixed-dimensional poromechanical models with frictional contact at matrix–fracture interfaces

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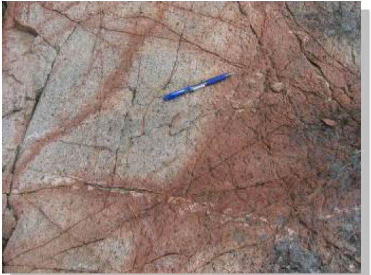
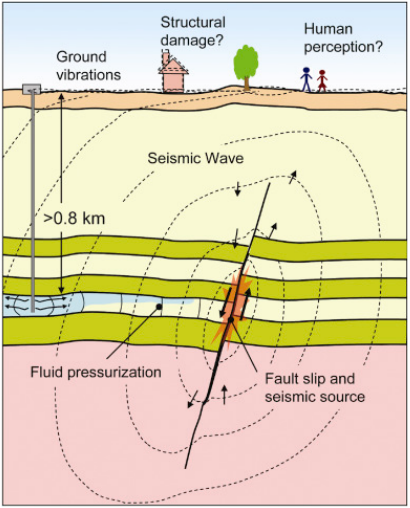
Fractured/faulted porous media: multiple scales (figures from J. R. de Dreuzy, Geosciences Rennes and Inria)

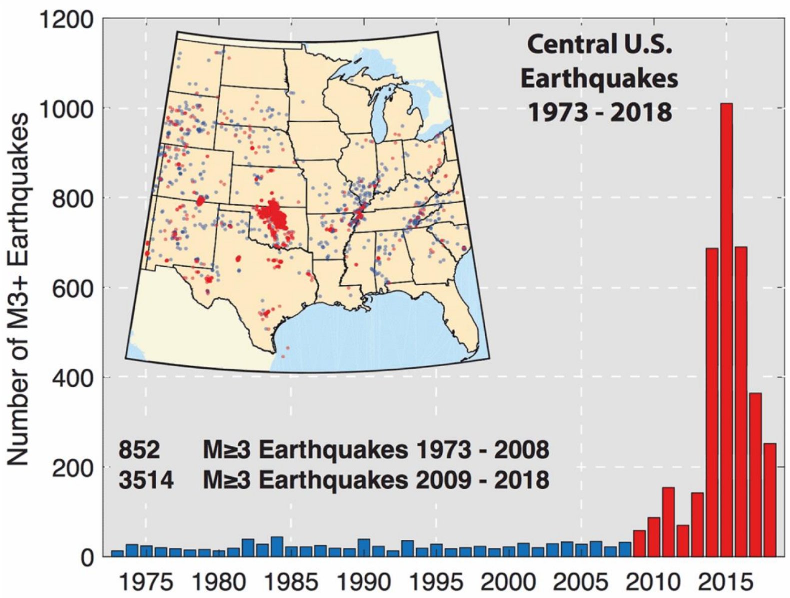


Fractured/faulted poro-mechanical models: risks of fault reactivation in CO_2 storage



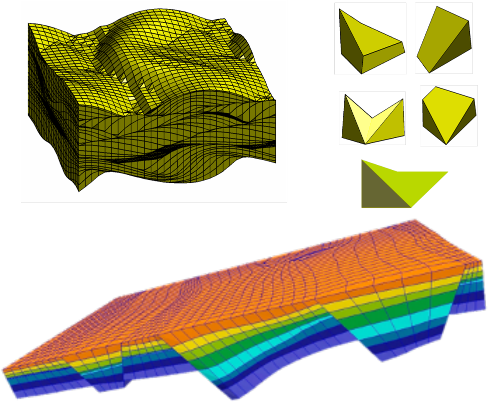
Rutqvist et al 2010

Induced seismicity



Typical meshes in geosciences

Corner Point Geometries (CPG)

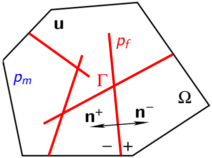


- Not adapted to Finite Element Methods (FEM) typically used in Mechanics
- Need for discretizations of contact mechanics adapted to **polyhedral** meshes

Outline

1. Contact-Mechanical model
2. Discretization on polyhedral meshes
3. Numerical validation
   - Contact-mechanics
   - Poromechanics
4. Fluid induced fault reactivation

- The matrix and fracture pressures p_m and p_f are fixed
- Isotropic linear poroelastic model in the matrix domain $\Omega \setminus \Gamma$



Mixed-dimensional geometry and unknowns

$$\left\{\begin{aligned} &-\mathrm{div}\Big(\mathbb{\sigma}^T(\mathbf{u}, p_m)\Big) = \mathbf{f},\\ &\mathbb{\sigma}^T(\mathbf{u}, p_m) = \mathbb{\sigma}(\mathbf{u}) - b\; p_m\; \mathbb{I},\\ &\mathbb{\sigma}(\mathbf{u}) = 2\mu\; \mathbb{\epsilon}(\mathbf{u}) + \lambda\; \mathrm{div}\,\mathbf{u}\; \mathbb{I}. \end{aligned}\right.$$

Fracture mechanical model on the fracture network Γ

Jumps : $[\![\mathbf{u}]\!] = \mathbf{u}^+ - \mathbf{u}^-, \ [\![\mathbf{u}]\!]_n = [\![\mathbf{u}]\!] \cdot \mathbf{n}^+, \ [\![\mathbf{u}]\!]_\tau = [\![\mathbf{u}]\!] - [\![\mathbf{u}]\!]_n \mathbf{n}^+,$

Surface Tractions: $\mathbf{T}^\pm = \sigma^T(\mathbf{u}, p_m)^\pm \mathbf{n}^\pm + p_f \mathbf{n}^\pm$

Law of Action and Reaction:

$$\mathbf{T}^+ + \mathbf{T}^- = \mathbf{0}$$

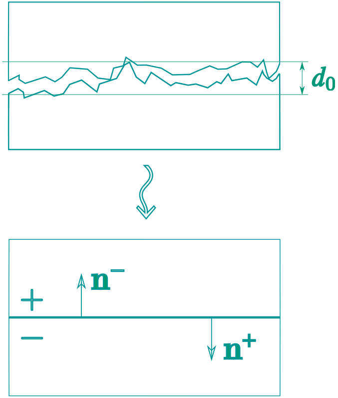
Non penetration conditions:

$$T_n^+ \leq 0, \quad [\![\mathbf{u}]\!]_n \leq 0, \quad [\![\mathbf{u}]\!]_n \ T_n^+ = 0$$

Coulomb friction conditions:

$$|\mathbf{T}_\tau^+| \leq -F \ T_n^+,$$

$$\mathbf{T}_\tau^+(\mathbf{u}) \cdot [\![\mathbf{u}]\!]_\tau - F \ T_n^+(\mathbf{u}) \, |[\![\mathbf{u}]\!]_\tau| = 0$$



Lagrange multiplier: $\lambda = -\mathbf{T}^+$

Dual cone of admissible Lagrange multipliers: given $\boldsymbol{\lambda} = (\lambda_{\mathbf{n}}, \boldsymbol{\lambda}_{\boldsymbol{\tau}})$

$$C_f(\lambda_{\mathbf{n}}) = \Big\{\boldsymbol{\mu} \in (H^{-1/2}(\Gamma))^d \;:\; \mu_{\mathbf{n}} \geq 0,\ |\boldsymbol{\mu}_{\boldsymbol{\tau}}| \leq F\lambda_{\mathbf{n}} \quad \text{(in a weak sense)}\Big\}.$$

Mixed variational inequality: $\mathbf{u} \in H_0^1(\Omega\backslash\overline{\Gamma})^d$, $\boldsymbol{\lambda} \in C_f(\lambda_{\mathbf{n}})$ such that

$$\int_\Omega \Big(\sigma(\mathbf{u}) : \varepsilon(\mathbf{v}) - b\; p_m \mathrm{div}(\mathbf{v})\Big) + \langle \boldsymbol{\lambda}, [\![\mathbf{v}]\!] \rangle_\Gamma + \int_\Gamma p_f\; [\![\mathbf{v}]\!]_{\mathbf{n}} = \int_\Omega \mathbf{f} \cdot \mathbf{v},$$

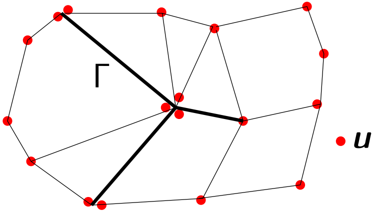
$$\langle \boldsymbol{\mu} - \boldsymbol{\lambda}, [\![\mathbf{u}]\!] \rangle_\Gamma \leq 0,$$

for all $\mathbf{v} \in H_0^1(\Omega\backslash\overline{\Gamma})^d$, $\boldsymbol{\mu} \in C_f(\lambda_{\mathbf{n}})$.

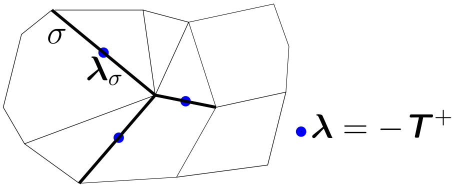
Polyhedral nodal discretization of the displacement field $\mathbf{u}$

- Virtual Element Method (VEM) [Beirao Da Veiga et al 2013]
- **Fully discrete approach** (nodal MFD, CDO, DDR)
  - local reconstruction operators from the space of discrete unknowns onto polynomial spaces.

Nodal displacement unknowns:



- Mixed formulation with nodal Lagrange multipliers [Wriggers et al 2016]
- Mixed formulation with face-wise constant Lagrange multipliers $\boldsymbol{\lambda} = -\mathbf{T}^+$
  - deal with fracture networks including intersections
  - face-wise contact conditions
  - preserve the contact dissipative properties



$$\mathbf{M}_{\mathcal{D}} = \left\{\boldsymbol{\lambda}_{\mathcal{D}} \in L^2(\Gamma)^d \;:\; \boldsymbol{\lambda}_{\mathcal{D}}(\mathbf{x}) = \boldsymbol{\lambda}_\sigma \;\forall \sigma \in \mathcal{F}_\Gamma, \forall\, \mathbf{x} \in \sigma\right\}.$$

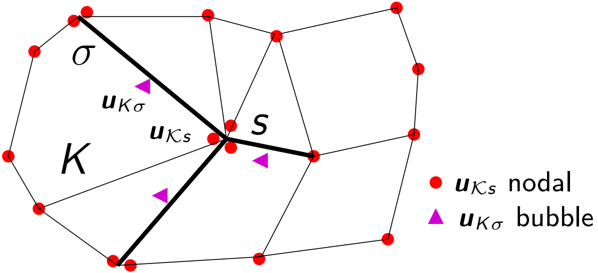
For $\boldsymbol{\lambda}_{\mathcal{D}} \in \mathbf{M}_{\mathcal{D}}$, we define the discrete dual cone of admissible Lagrange multipliers:

$$\mathbf{C}_{\mathcal{D}}\left(\lambda_{\mathcal{D},\mathbf{n}}\right) = \left\{\boldsymbol{\mu}_{\mathcal{D}} = \left(\mu_{\mathcal{D},\mathbf{n}}, \boldsymbol{\mu}_{\mathcal{D},\boldsymbol{\tau}}\right) \in \mathbf{M}_{\mathcal{D}} \;:\; \mu_{\mathcal{D},\mathbf{n}} \geq 0,\; |\boldsymbol{\mu}_{\mathcal{D},\boldsymbol{\tau}}| \leq F\lambda_{\mathcal{D},\mathbf{n}}\right\}.$$

Stabilization of the Lagrange multiplier

A stabilization is required to avoid spurious Lagrange multiplier modes

- Enrichment of the displacement space
  - $\mathbb{P}^1$-bubble FEM [Renard et al 2003]
  - **In this work: polytopal bubble stabilisation**



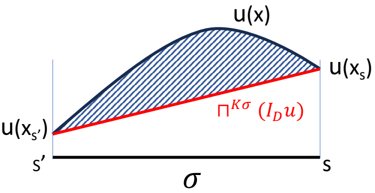
Vector space of discrete displacement unknowns:

$$\mathbf{U}_{\mathcal{D}} = \left\{ \mathbf{v}_{\mathcal{D}} = \left((\mathbf{v}_{\mathcal{K}s})_{\mathcal{K}s \in \overline{\mathcal{M}}_s, s \in \mathcal{V}}, (\mathbf{v}_{K\sigma})_{\sigma \in \mathcal{F}^+_{\Gamma,K}, \ K \in \mathcal{M}} \right) \ : \ \mathbf{v}_{\mathcal{K}s} \in \mathbb{R}^d \ , \ \mathbf{v}_{K\sigma} \in \mathbb{R}^d \right\}$$

Interpolation operator

$$\mathcal{I}_{\mathcal{D}} : C^0(\Omega \setminus \overline{\Gamma})^d \to \mathbf{U}_{\mathcal{D}}$$

$$\begin{cases} (\mathcal{I}_{\mathcal{D}}\mathbf{u})_{\mathcal{K}s} = \mathbf{u}_{|_K}(\mathbf{x}_s), \\ (\mathcal{I}_{\mathcal{D}}\mathbf{u})_{K\sigma} = \frac{1}{|\sigma|}\int_\sigma (\gamma^{K\sigma}\mathbf{u} - \Pi^{K\sigma}(\mathcal{I}_{\mathcal{D}}\mathbf{u})). \end{cases}$$



- $\gamma^{K\sigma}$ is the trace operator on σ from the K side
- $\Pi^{K\sigma}$ is the face linear reconstruction operator depending only on the nodal degrees of freedom

Cell gradient and function reconstruction operators:

- $\nabla^K : \mathbf{U}_{\mathcal{D}} \to (\mathbb{P}^0(K))^{d\times d}$
- $\Pi^K : \mathbf{U}_{\mathcal{D}} \to (\mathbb{P}^1(K))^d$

Fracture face mean displacement jump:

- $[\![\ ]\!]_\sigma : \mathbf{U}_{\mathcal{D}} \to \mathbb{P}^0(\sigma)^d$

Global piecewise reconstruction operators:

- $(\varepsilon_{\mathcal{D}}(\mathbf{u}_{\mathcal{D}}))|_K = \frac{1}{2}(\nabla^K \mathbf{u}_{\mathcal{D}} + {}^t\nabla^K \mathbf{u}_{\mathcal{D}})$
- $\mathrm{div}_{\mathcal{D}} = \mathrm{tr}(\varepsilon_{\mathcal{D}}), \quad \sigma_{\mathcal{D}} = 2\mu\, \varepsilon_{\mathcal{D}} + \lambda\, \mathrm{div}_{\mathcal{D}}\, \mathbb{I}$
- $(\Pi_{\mathcal{D}} \mathbf{u}_{\mathcal{D}})|_K = \Pi^K \mathbf{u}_{\mathcal{D}}$
- $([\![\mathbf{u}_{\mathcal{D}}]\!]_{\mathcal{D}})|_\sigma = [\![\mathbf{u}_{\mathcal{D}}]\!]_\sigma$

Discrete mixed variational formulation

Find $(\mathbf{u}_{\mathcal{D}}, \boldsymbol{\lambda}_{\mathcal{D}}) \in \mathbf{U}^0_{\mathcal{D}} \times \mathbf{C}_{\mathcal{D}}(\lambda_{\mathcal{D},\mathbf{n}})$, such that:

$$\left\{\begin{array}{l}
\displaystyle\int_\Omega \sigma_{\mathcal{D}}(\mathbf{u}_{\mathcal{D}}) : \varepsilon_{\mathcal{D}}(\mathbf{v}_{\mathcal{D}}) + S_{\mu,\lambda,\mathcal{D}}(\mathbf{u}_{\mathcal{D}}, \mathbf{v}_{\mathcal{D}}) - \int_\Omega b\, p_m \,\mathrm{div}_{\mathcal{D}} \mathbf{v}_{\mathcal{D}} \\
\displaystyle + \int_\Gamma p_f [\![\mathbf{v}_{\mathcal{D}}]\!]_{\mathcal{D},\mathbf{n}} + \int_\Gamma \boldsymbol{\lambda}_{\mathcal{D}} \cdot [\![\mathbf{v}_{\mathcal{D}}]\!]_{\mathcal{D}} = \sum_{K \in \mathcal{M}} \frac{1}{|K|} \int_K \mathbf{f} \cdot \int_K \Pi_{\mathcal{D}} \mathbf{v}_{\mathcal{D}}, \\
\\
\displaystyle\int_\Gamma (\boldsymbol{\mu}_{\mathcal{D}} - \boldsymbol{\lambda}_{\mathcal{D}}) \cdot [\![\mathbf{u}_{\mathcal{D}}]\!]_{\mathcal{D}} \leq 0,
\end{array}\right.$$

for all $(\mathbf{v}_{\mathcal{D}}, \boldsymbol{\mu}_{\mathcal{D}}) \in \mathbf{U}^0_{\mathcal{D}} \times \mathbf{C}_{\mathcal{D}}(\lambda_{\mathcal{D},\mathbf{n}})$.

The variational inequality can be reformulated by local to each fracture face equations:

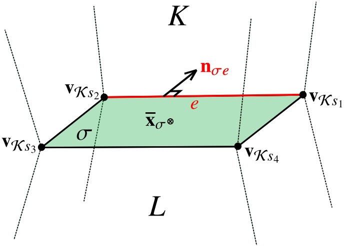
$$\left\{\begin{array}{l}
\lambda_{\sigma,\mathbf{n}} = \left[\lambda_{\sigma,\mathbf{n}} + \beta_{\sigma,\mathbf{n}} [\![\mathbf{u}_{\mathcal{D}}]\!]_{\sigma,\mathbf{n}}\right]_{\mathbb{R}^+} \\
\boldsymbol{\lambda}_{\sigma,\boldsymbol{\tau}} = \left[\boldsymbol{\lambda}_{\sigma,\boldsymbol{\tau}} + \beta_{\sigma,\boldsymbol{\tau}} [\![\mathbf{u}_{\mathcal{D}}]\!]_{\sigma,\boldsymbol{\tau}}\right]_{F\lambda_{\sigma,\mathbf{n}}}
\end{array}\right.$$

with $[x]_{\mathbb{R}^+} = \max\{0, x\}$ and $[\mathbf{x}]_\alpha = \left\{\begin{array}{ll} \mathbf{x} & \text{if } |\mathbf{x}| \leq \alpha, \\ \alpha \dfrac{\mathbf{x}}{|\mathbf{x}|} & \text{otherwise,} \end{array}\right.$ $\quad \beta_{\mathcal{D},\mathbf{n}} > 0, \quad \beta_{\mathcal{D},\boldsymbol{\tau}} > 0.$

Face mean value reconstruction:

$$\overline{\mathbf{v}}_{K\sigma} = \sum_{s\in\mathcal{V}_\sigma} \omega_s^\sigma \mathbf{v}_{\mathcal{K}s}$$

with the face center of mass $\overline{\mathbf{x}}_\sigma = \sum_{s\in\mathcal{V}_\sigma} \omega_s^\sigma \mathbf{x}_s$.



Face average displacement jump operator:

$$[\![\ ]\!]_\sigma : \mathbf{U}_\mathcal{D} \to \mathbb{P}^0(\sigma)^d$$

$$[\![\mathbf{v}_\mathcal{D}]\!]_\sigma = \overline{\mathbf{v}}_{K\sigma} - \overline{\mathbf{v}}_{L\sigma} + \mathbf{v}_{K\sigma}.$$

The gradient reconstruction operator:

$$\nabla^K : \mathbf{U}_{\mathcal{D}} \to (\mathbb{P}^0(K))^{d\times d}$$

$$\nabla^K \mathbf{v}_{\mathcal{D}} = \frac{1}{|K|} \sum_{\sigma\in\mathcal{F}^+_{\Gamma,K}} |\sigma| \mathbf{v}_{K\sigma} \otimes \mathbf{n}_{K\sigma} + \frac{1}{|K|} \sum_{\sigma\in\mathcal{F}_K} |\sigma| \overline{\mathbf{v}}_{K\sigma} \otimes \mathbf{n}_{K\sigma}.$$

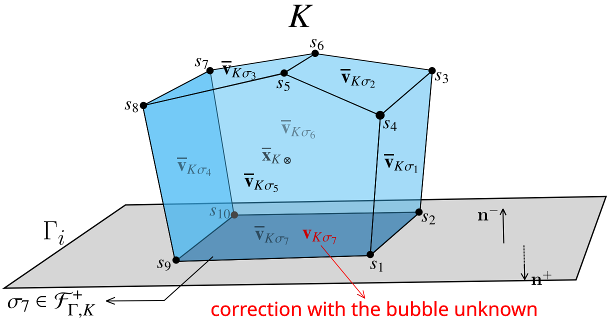


Figure: Nodal and bubble unknowns in a cell K

Linear function reconstruction operator:

$$\Pi^K : \mathbf{U}_{\mathcal{D}} \to (\mathbb{P}^1(K))^d$$

$$\Pi^K \mathbf{v}_{\mathcal{D}}(\mathbf{x}) = \nabla^K \mathbf{v}_{\mathcal{D}} \, (\mathbf{x} - \bar{\mathbf{x}}_K) + \bar{\mathbf{v}}_K,$$

with

$$\bar{\mathbf{v}}_K = \sum_{s \in \mathcal{V}_K} \omega_s^K \mathbf{v}_{\mathcal{K}s}$$

and the cell center of mass $\bar{\mathbf{x}}_K = \sum_{s \in \mathcal{V}_K} \omega_s^K \mathbf{x}_s.$

$S_{\mu,\lambda,\mathcal{D}}$ is the scaled stabilisation bilinear form defined by:

$$S_{\mu,\lambda,\mathcal{D}}(\mathbf{u}_{\mathcal{D}},\mathbf{v}_{\mathcal{D}}) = \sum_{K\in\mathcal{M}} h_K^{d-2}(2\mu_K+\lambda_K)S_K(\mathbf{u}_{\mathcal{D}},\mathbf{v}_D),$$

with

$$S_K(\mathbf{u}_{\mathcal{D}},\mathbf{v}_{\mathcal{D}}) = \sum_{s\in\mathcal{V}_K}(\mathbf{u}_{\mathcal{K}s} - \Pi^K\mathbf{u}_{\mathcal{D}}(\mathbf{x}_s))\cdot(\mathbf{v}_{\mathcal{K}s} - \Pi^K\mathbf{v}_{\mathcal{D}}(\mathbf{x}_s)) + \sum_{\sigma\in\mathcal{F}^+_{\Gamma,K}} \mathbf{u}_{K\sigma}\cdot\mathbf{v}_{K\sigma},$$

such that

$$S_K(\mathcal{I}_{\mathcal{D}}\mathbf{q},\mathbf{v}_{\mathcal{D}}) = S_K(\mathbf{u}_{\mathcal{D}},\mathcal{I}_{\mathcal{D}}\mathbf{q}) = 0$$

for all $\mathbf{q}\in\mathbb{P}^1(K)$.

Error estimate for Tresca friction

Let $(\mathbf{u},\lambda)$ be the exact solution and assume that $\mathbf{u} \in H^2(\mathcal{M})$ and $\lambda \in H^1(\mathcal{F}_\Gamma)$. Then the discrete solution $(\mathbf{u}_\mathcal{D},\lambda_\mathcal{D})$ satisfies the following error estimate:

$$\|\nabla^\mathcal{D}\mathbf{u}_\mathcal{D} - \nabla\mathbf{u}\|_{L^2(\Omega\setminus\overline{\Gamma})} + \|\lambda_\mathcal{D} - \lambda\|_{-1/2,\Gamma} \lesssim h_\mathcal{D}(|\lambda|_{H^1(\mathcal{F}_\Gamma)} + |\mathbf{u}|_{H^2(\mathcal{M})}).$$

The proof is mainly based on the **discrete inf-sup condition**:

$$\sup_{\mathbf{v}_\mathcal{D}\in\mathbf{U}^0_\mathcal{D}} \frac{\int_\Gamma \lambda_\mathcal{D}\cdot[\![\mathbf{v}_\mathcal{D}]\!]_\mathcal{D}}{\|\mathbf{v}_\mathcal{D}\|_{1,\mathcal{D}}} \gtrsim \|\lambda_\mathcal{D}\|_{-1/2,\Gamma} \qquad \forall\lambda_\mathcal{D}\in\mathbf{M}_\mathcal{D}.$$

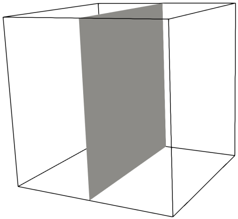
with $\|\mathbf{v}_\mathcal{D}\|_{1,\mathcal{D}} := \left(\sum_{K\in\mathcal{M}}(\|\nabla^K\mathbf{v}_\mathcal{D}\|^2_{L^2(K)} + S_K(\mathbf{v}_\mathcal{D},\mathbf{v}_\mathcal{D}))\right)^{1/2}.$

and the **discrete Korn inequality**:

$$\|\mathbf{v}_\mathcal{D}\|^2_{1,\mathcal{D}} \lesssim \|\mathbb{\epsilon}_\mathcal{D}(\mathbf{v}_\mathcal{D})\|^2_{L^2(\Omega\setminus\overline{\Gamma})} + \sum_{K\in\mathcal{M}} S_K(\mathbf{v}_\mathcal{D},\mathbf{v}_\mathcal{D}) \qquad \forall\mathbf{v}_\mathcal{D}\in\mathbf{U}^0_\mathcal{D}.$$

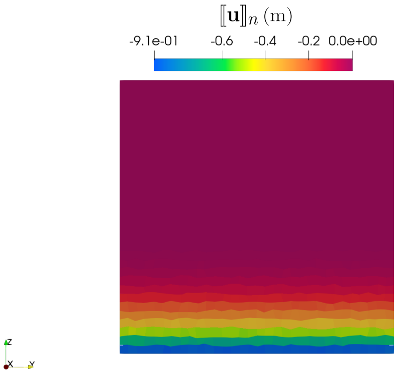
Frictionless contact mechanical model:

$$\begin{cases} -\mathrm{div}\sigma(\mathbf{u}) = \mathbf{f} & \text{on } \Omega\backslash\overline{\Gamma} \\ \sigma(\mathbf{u}) = 2\mu\ \epsilon(\mathbf{u}) + \lambda\ \mathrm{div}\,\mathbf{u}\ \mathbb{I} & \text{on } \Omega\backslash\overline{\Gamma} \\ \mathbf{T}^+ + \mathbf{T}^- = \mathbf{0} & \text{on } \Gamma \\ T_n \leq 0,\ [\![\mathbf{u}]\!]_n \leq 0,\ [\![\mathbf{u}]\!]_n\ T_n = 0 & \text{on } \Gamma \\ \mathbf{T}_\tau = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

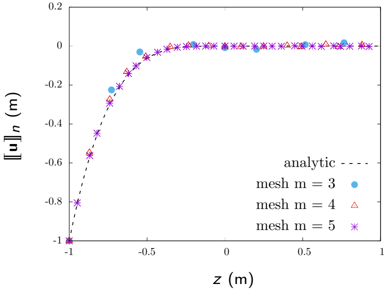


Analytical solution:

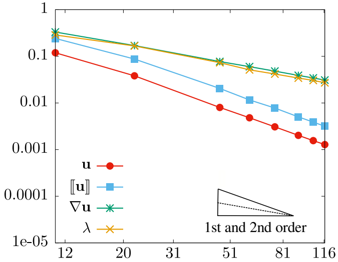
$$\mathbf{u}(x,y,z) = \begin{cases} \begin{pmatrix} g(x,y)p(z) \\ p(z) \\ x^2 p(z) \end{pmatrix} & \text{if } z \geq 0, \\ \begin{pmatrix} h(x)p^+(z) \\ h(x)\,(p^+(z))' \\ -\int_0^x h(\xi)\mathrm{d}\xi\,(p^+(z))' \end{pmatrix} & \text{if } z < 0,\ x < 0, \\ \begin{pmatrix} h(x)p^-(z) \\ h(x)\,(p^-(z))' \\ -\int_0^x h(\xi)\mathrm{d}\xi\,(p^-(z))' \end{pmatrix} & \text{if } z < 0,\ x \geq 0, \end{cases} \quad \text{with} \quad \begin{cases} g(x,y) = -\sin(\frac{\pi x}{2})\cos(\frac{\pi y}{2}) \\ p(z) = z^2 \\ h(x) = \cos(\frac{\pi x}{2}) \\ p^+(z) = z^4 \\ p^-(z) = 2z^4 \end{cases}$$



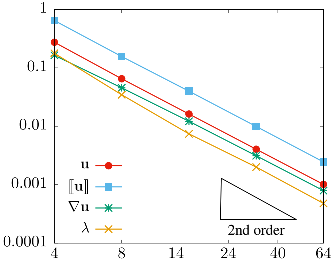
$[\![\mathbf{u}]\!]_n$ (m)
-9.1e-01
-0.6
-0.4
-0.2
0.0e+00
Z
X
Y



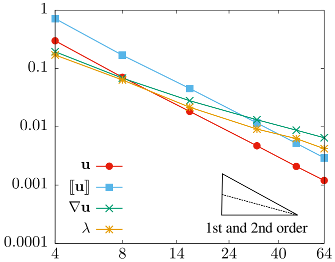
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
$[\![\mathbf{u}]\!]_n$ (m)
z (m)
analytic
mesh m = 3
mesh m = 4
mesh m = 5



(a)



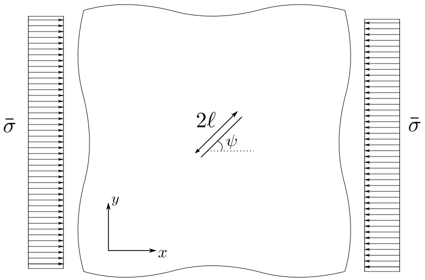
(b)

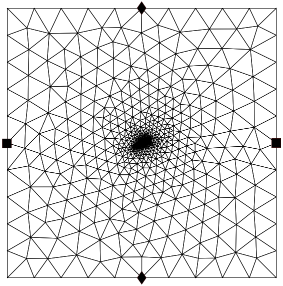


(c)

Figure: Error and convergence rates obtained with the VEM $\mathbb{P}^1$-bubble method: Tetrahedral mesh (a), cartesian mesh (b), polytopal mesh (c).

Single crack under compression



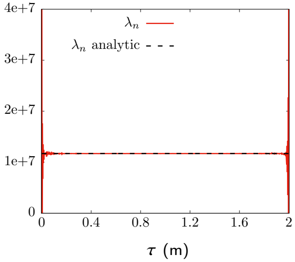


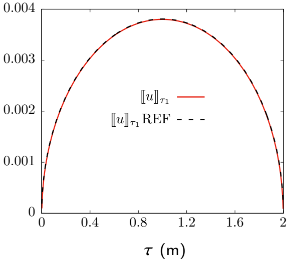
♦: $u_x = 0$, ■: $u_y = 0$

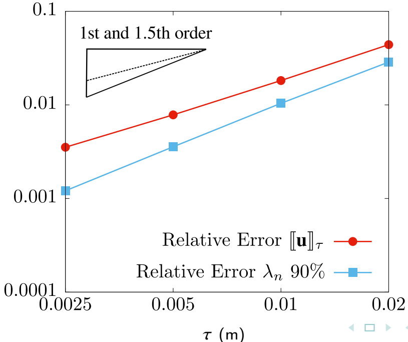
$$|\,[\![\bar{\mathbf{u}}]\!]_\tau(\tau)\,| = \frac{4(1-\nu)}{E}(\bar{\sigma}\sin\psi(\cos\psi - F\sin\psi))\sqrt{\ell^2 - (\ell^2 - \tau^2)},$$

$$\bar{\lambda}_n(\tau) = \bar{\sigma}\sin^2\psi, \qquad 0 \le \tau \le 2\ell$$

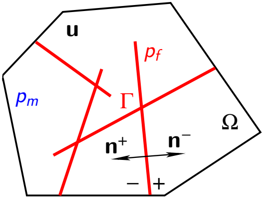
Single crack under compression







$$\begin{cases} \partial_t \phi_m + \operatorname{div} \mathbf{V}_m = h_m & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma}, \\ \mathbf{V}_m = -\frac{\mathbb{K}_m}{\eta}\nabla p_m & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma}, \\ \partial_t \mathrm{d}_f + \operatorname{div}_\tau \mathbf{V}_f - [\![\mathbf{V}_m]\!]_n = h_f & \text{on } (0,T)\times\Gamma, \\ \mathbf{V}_f = \frac{C_f(\mathrm{d}_f)}{\eta}\nabla_\tau p_f, & \text{on } (0,T)\times\Gamma, \\ \mathbf{V}_m^\pm\cdot\mathbf{n}^\pm = T_f(\mathrm{d}_f)(\gamma^\pm p_m - p_f) & \text{on } (0,T)\times\Gamma, \end{cases}$$



with the following coupling laws

$$\begin{cases} \partial_t \phi_m = b\operatorname{div}(\partial_t \mathbf{u}) + \frac{1}{M}\partial_t p_m & \text{on } (0,T)\times\Omega\backslash\overline{\Gamma}, \\ \mathrm{d}_f = \mathrm{d}_f^c - [\![\mathbf{u}]\!]_{\mathbf{n}} & \text{on } (0,T)\times\Gamma, \end{cases}$$

Hybrid Finite Volume (HFV) discretisation for the Darcy flow model [Brenner et al 2016]

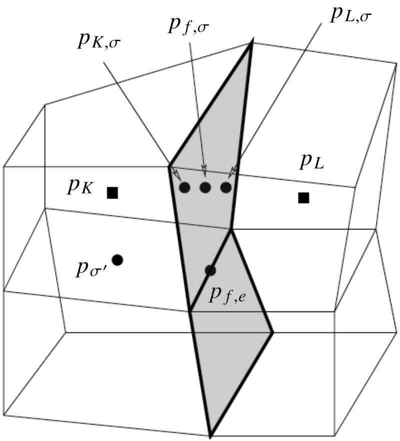


Figure: Pressure unknowns for the HFV scheme with discontinuous pressure

Any solution $\left(p_{\mathcal{D}}^n, \mathbf{u}_{\mathcal{D}}^n, \lambda_D^n)\right) \in X_{\mathcal{D}}^0 \times \mathbf{U}_{\mathcal{D}}^0 \times \mathbf{C}_{\mathcal{D}}(\lambda_{\mathcal{D},\mathbf{n}}^n)$ for $n = 1, \cdots, N$ of the fully coupled scheme satisfies the following discrete energy estimates:

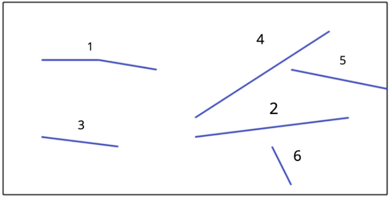
$$\delta_t^n \int_\Omega \frac{1}{2}\left(\sigma_{\mathcal{D}}(\mathbf{u}_{\mathcal{D}}) : \varepsilon_{\mathcal{D}}(\mathbf{u}_{\mathcal{D}}) + S_{\mu,\lambda,\mathcal{D}}(\mathbf{u}_{\mathcal{D}}, \mathbf{u}_{\mathcal{D}}) + \frac{1}{M}|\Pi_{\mathcal{D}_m} p_{\mathcal{D}_m}|^2\right) + \int_\Gamma F\lambda_{\mathcal{D},\mathbf{n}}^n |[\![\delta_t^n \mathbf{u}_{\mathcal{D}}]\!]_{\mathcal{D},\tau}|$$

$$+ \int_\Omega \frac{\mathbb{K}_m}{\eta} \nabla_{\mathcal{D}_m} p_{\mathcal{D}_m}^n \cdot \nabla_{\mathcal{D}_m} p_{\mathcal{D}_m}^n + \int_\Gamma \frac{C_{f,\mathcal{D}}^{n-1}}{\eta} |\nabla_{\mathcal{D}_f} p_{\mathcal{D}_f}^n|^2 + \sum_{\mathfrak{a}\in\{+,-\}} \int_\Gamma \Lambda_{f,\mathcal{D}}^{n-1} ([\![p_{\mathcal{D}}^n]\!]_{\mathcal{D}}^{\mathfrak{a}})^2$$

$$\leq \int_\Omega h_m \Pi_{\mathcal{D}_m} p_{\mathcal{D}_m}^n + \int_\Gamma h_f \nabla_{\mathcal{D}_f} p_{\mathcal{D}_f}^n + \sum_{K\in\mathcal{M}} \int_K \mathbf{f}_K^n \cdot \Pi_{\mathcal{D}} \delta_t^n \mathbf{u}_{\mathcal{D}}.$$

Thanks to the dissipative property of the contact term:

$$\int_\Gamma \lambda_{\mathcal{D}}^n \cdot [\![\delta_t^n \mathbf{u}_{\mathcal{D}}]\!]_{\mathcal{D}} \geq \int_\Gamma F\lambda_{\mathcal{D},\mathbf{n}}^n |[\![\delta_t^n \mathbf{u}_{\mathcal{D}}]\!]_{\mathcal{D},\tau}| \geq 0.$$



Anisotropic permeability tensor:

$$\mathbb{K}_m = 10^{-15}\left(\mathbf{e}_x \otimes \mathbf{e}_x + \frac{1}{2}\,\mathbf{e}_y \otimes \mathbf{e}_y\right)$$

Fracture aperture in contact state:

$$d_f^c(\mathbf{x}) = 10^{-4}\,\frac{\sqrt{\arctan(aD_i(\mathbf{x}))}}{\sqrt{\arctan(a\ell_i)}}, \quad i \in \{1,\dots,6\}$$

$$F = 0.5,\ b = 0.5,\ E = 10 \text{ GPa},\ \nu = 0.2$$

No analytical solution available

$\Rightarrow$ Compute reference solution on fine mesh

[Acknowledgement: E. Keilegavlen (Bergen)]

Initial conditions

Initial pressures $p_m^0 = p_f^0 = 10^5$ Pa

Boundary conditions

Mechanics

Top boundary:

$$\mathrm{u}(t,\mathrm{x}) = \begin{cases} {}^t[0.005\,\mathrm{m}, -0.002\,\mathrm{m}]\,4t/T & \text{if } t \leq T/4 \\ {}^t[0.005\,\mathrm{m}, -0.002\,\mathrm{m}] & \text{otherwise} \end{cases}$$

Bottom boundary: $\mathbf{u}(t,\mathbf{x}) \equiv \mathbf{0}$

Left and right boundaries: $\sigma^T(t,\mathbf{x})\mathbf{n}(\mathbf{x}) \equiv \mathbf{0}$

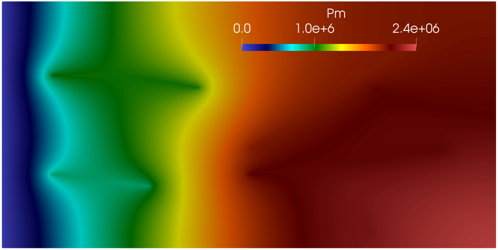
Flow

Left boundary: $p_m(t,\mathbf{x}) \equiv p_m^0 = 10^5$ Pa

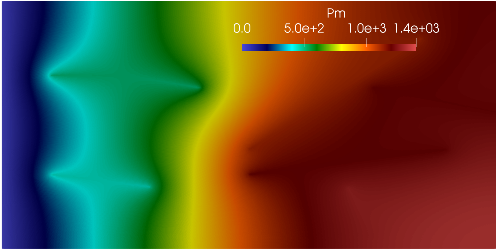
All other boundaries: impervious

Matrix over pressures

End of Stage 1 at $t = \frac{T}{4}$



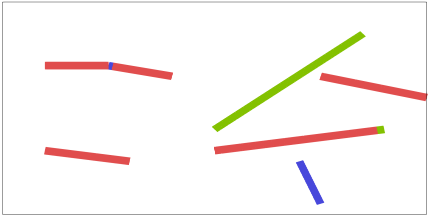
End of Stage 2 at $t = T$



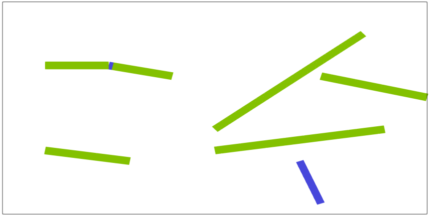
Contact state along the fractures

- $-T_n = -(\sigma(\mathbf{u}) \cdot \mathbf{n}) \cdot \mathbf{n} - (1-b)p_f \quad and \quad |\mathbf{T}_\tau| \leq -F \ T_n$

End of Stage 1 at $t = \frac{T}{4}$



End of Stage 2 at $t = T$



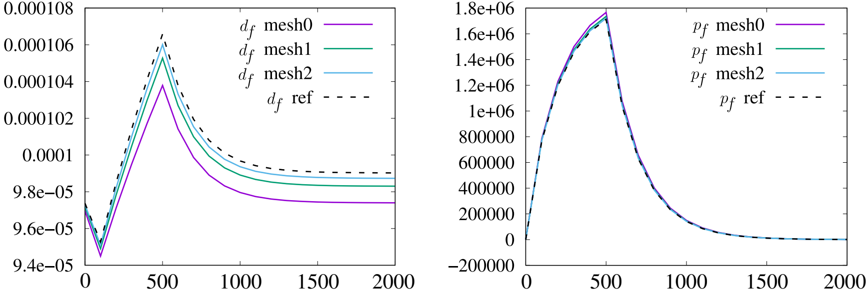
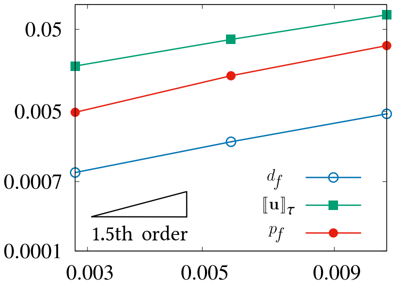


Figure: Mean aperture and mean pressure in fractures as a function of time.



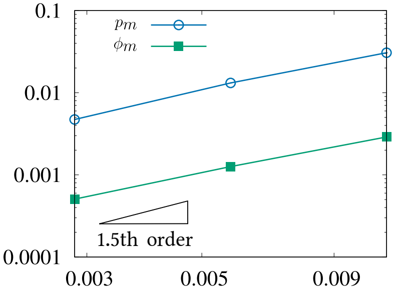


Figure: Relative L_2 error between the current and reference solution

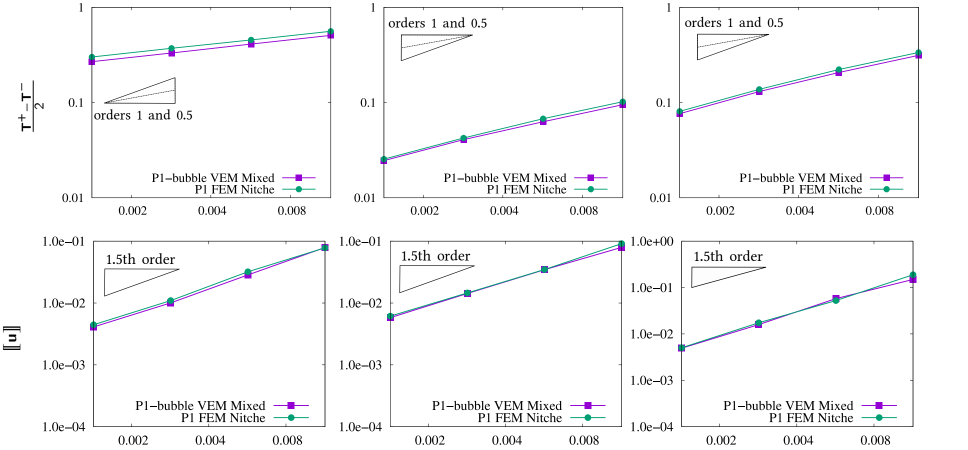
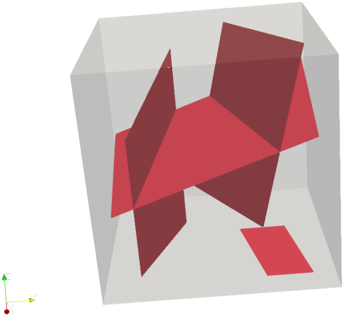


Figure: Relative L_2 error, as a function of the size of the largest fracture face, between the current and reference solutions in terms of $(\mathbf{T}^+ - \mathbf{T}^-)/2$ (top) and $[\![\mathbf{u}]\!]$ (bottom) along fractures 1,2 and 3 from left to right: Mixed $\mathbb{P}^1$-bubble VEM - $\mathbb{P}^0$ vs Nitsche $\mathbb{P}^1$ FEM.

3D DFM poromechanical test case



Isotropic permeability tensor:

$$\mathbb{K}_m = 10^{-14}\ \mathbb{I}\ (\mathrm{m}^2)$$

Fracture aperture in contact state:

$$d_f^c(\mathbf{x}) = 10^{-3}\ \mathrm{m}$$

$F = 0.5,\ b = 0.5,\ E = 10$ GPa, $\nu = 0.2$

Initial conditions

Initial pressures $p_m^0 = p_f^0 = 10^5$ Pa

Boundary conditions

Mechanics

Top boundary:

$$\mathrm{u}(t,\mathrm{x}) = \begin{cases} {}^t[0.005\,\mathrm{m}, 0.002\,\mathrm{m}, -0.002\,\mathrm{m}]\,2t/T & \text{if } t \le T/2 \\ {}^t[0.005\,\mathrm{m}, 0.002\,\mathrm{m}, -0.002\,\mathrm{m}] & \text{otherwise} \end{cases}$$

Bottom boundary: $\mathbf{u}(t,\mathbf{x}) \equiv \mathbf{0}$

Lateral boundaries: $\sigma^T(t,\mathbf{x})\mathbf{n}(\mathbf{x}) \equiv \mathbf{0}$

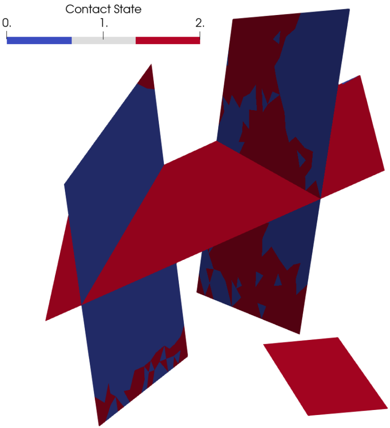
Flow

Boundary $y = 0$ and $y = 1$: $p_m(t,\mathbf{x}) \equiv p_m^0 = 10^5$ Pa

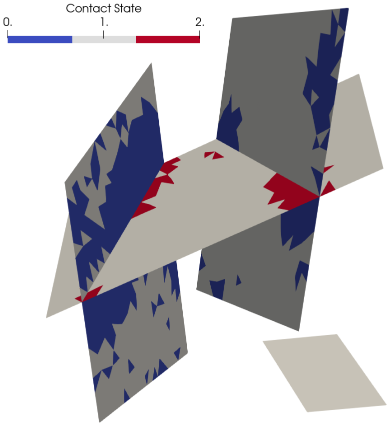
All other boundaries: impervious

3D DFM poromechanical test case: contact state

$t = T/2$



$t = T$



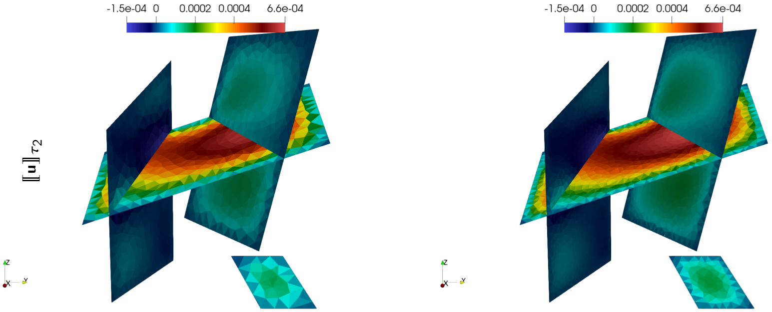


Figure: The τ_2 component of the tangential jump with the 47k cells mesh (left) and the 127k cells mesh (right), obtained at final time.

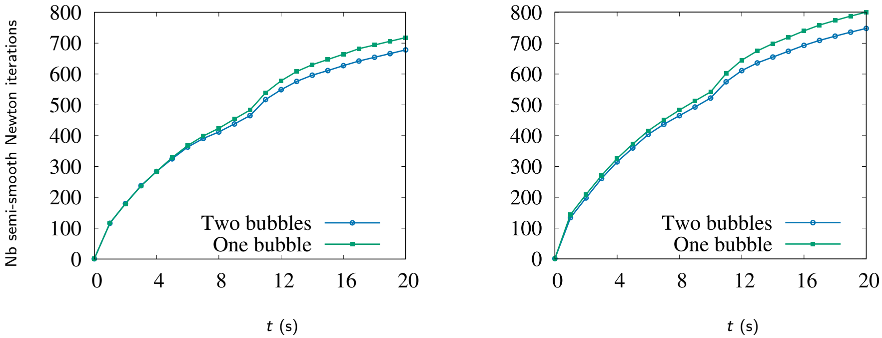
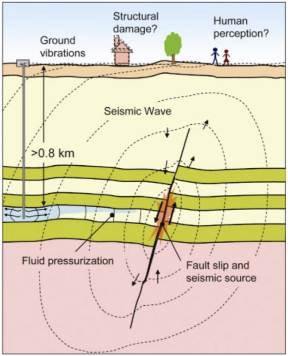
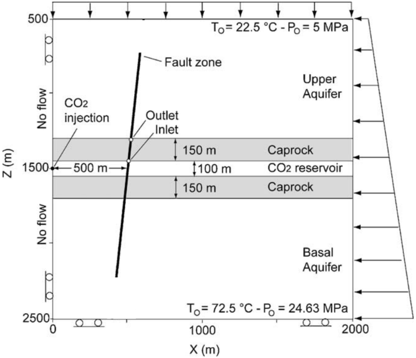


Figure: Total number of semi-smooth Newton iterations for the contact-mechanical model as a function of time, with both one-sided and two-sided bubbles and for both meshes with 47k cells (left) and 127k cells (right).

Fault reactivation by fluid injection

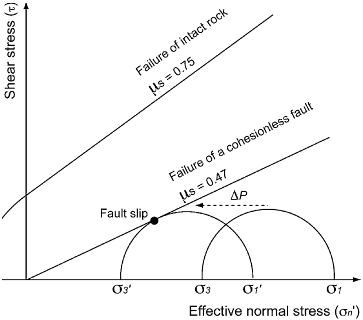


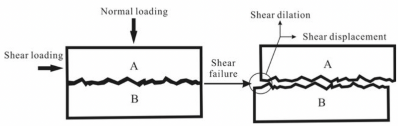
F. Cappa, J. Rutqvist 2010



$$
\begin{aligned}
\mathbb{K}_{reservoir} &= 10^{-13}\,\mathbb{I}\,m^2,\\
\mathbb{K}_{caprock} &= 10^{-19}\,\mathbb{I}\,m^2,\\
\mathbb{K}_{aquifer} &= 10^{-15}\,\mathbb{I}\,m^2.
\end{aligned}
$$

Fault reactivation by fluid injection

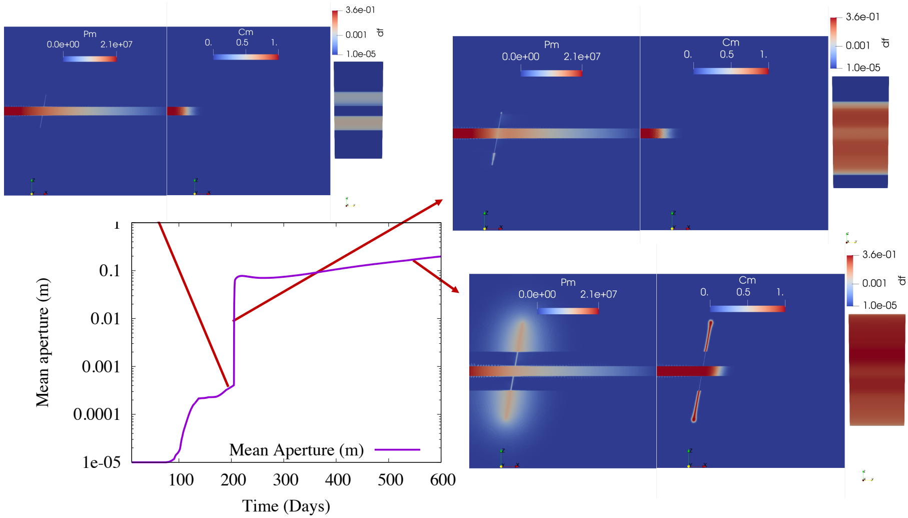




$$d_f = d_f^c - [\![\mathbf{u}]\!]_{\mathbf{n}} + \tan(\psi)|[\![\mathbf{u} - \mathbf{u}^0]\!]_{\boldsymbol{\tau}}|$$

$$d_f^c = 10^{-5} \text{ m}$$

Fault reactivation by fluid injection



Conclusions and Perspectives

- Conclusions
  - Extension to polytopal framework of face bubble stabilisation for contact-mechanics
  - Energy stable discretisation of mixed-dimensional poro-mechanical models
  - Application to the simulation of fluid-induced fault reactivation

- Perspectives
  - Coupling algorithms
  - Linear solvers

  - Nitsche's formulation: see poster of Mohamed Laaziri
  - Higher order polytopal method

  - Thermo-Hydro-Mechanics
  - Two-phase flows
  - Dynamic friction (seismic slip)

Thank you for your attention.